6 Asymptotic radiative transfer

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6.1 Introduction

Light propagation in turbid media such as the atmosphere and the ocean is usually studied in the framework of radiative transfer theory. In particular, solutions of the integro-differential radiative transfer equation (RTE) are analysed for media having different shapes and internal microstructure. A number of numerical and analytical techniques have been developed to date (Chandrasekhar, 1950; Sobolev, 1975; van de Hulst, 1980; Nakajima and Tanaka, 1988; Thomas and Stamnes, 1999; Siewert, 2000; Liou, 2002).

A popular technique for a numerical algorithm is based on the iteration approach (Liou, 2002). Then the single scattering solution is used to obtain the result for the first iteration. The obtained solution is substituted in the integral term of RTE to find the next iteration and the procedure is repeated until the convergence is reached. This technique is of a special importance for studies of radiative transfer in turbid media with complex shapes (Nikolaeva *et al.*, 2005). However, the iteration technique requires quite large computational time for optically thick media.

Therefore, yet another approach has been developed to treat a special case of optically thick turbid media. In particular, this technique allows us to represent the turbid layer reflectance as a combination of the reflectance for the case of a semi-infinite turbid medium minus the correction term, which accounts for the finite thickness of a layer under consideration. The correspondening asymptotic radiative transfer theory (ARTT) has been developed by Germogenova (1961), Rozenberg (1962), Sobolev (1968, 1975), van de Hulst (1968a, 1968b), Minin (1988), Zege *et al.* (1991), and Yanovitskij (1997).

The task of this chapter is to make a review of recent results obtained in the framework of ARTT. We hope that this work will stimulate the application of the theory to the solution of various applied problems related to light propagation in turbid media.

6.2 Radiative transfer equation

Light scattering by a single macroscopic particle can be studied in the framework of electrodynamics of continuous media. The same applies to clusters of particles or scattering volumes, where multiple light scattering does not play an important role. This is not the case for optically thick light scattering media. Here multiple scattering dominates the registered signal. Therefore, generally speaking, techniques of multiple wave scattering should be used in this case. However, they are quite complex and do not always lead to results, which can be used as a base for the numerical algorithm.

Moreover, electromagnetic fields $\boldsymbol{E}$ cannot be measured in the optical range. This is mostly due to their high oscillations ($\approx 10^{15}$ oscillations per second). Clearly, a measuring device makes temporal and spatial averaging of the registered signal. Also optical instruments measure quantities quadratic with respect to the field. This is similar to quantum mechanics, where the amplitude ψ is the main notion of the theory, but it is $|\psi|^2$, which is measured.

Therefore, it is of importance to formulate multiple light scattering theory, not in terms of field vectors but in terms of quadratic values, which can be easily measured. The Stokes-vector parameter $\boldsymbol{I}$ with components $\mathrm{I}, \mathrm{Q}, \mathrm{U}, \mathrm{V}$ (Stokes, 1852) is usually used in this case. Of course, this leads to the omission of a number of theoretical details (e.g., related to the phase effects). However, such an approach allows an interpretation of most optical measurements. Also light beams having the same values of $\mathrm{I}, \mathrm{Q}, \mathrm{U}, \mathrm{V}$ (but in principle different values of $\boldsymbol{E}$) cannot be distinguished by optical instruments, which measure quadratic values. Therefore, the main point is to force multiple light scattering theory to deal with intensities rather than fields from the very beginning. Then we do not need to make corresponding averaging procedures at the end of calculations to bring calculated values into correspondence to the measured ones. The main aim of this section is to introduce an equation, which governs the transformation of the light intensity due to multiple scattering processes in turbid media.

For the sake of simplicity, we consider the transformation of light intensity and ignore other components of the Stokes vector. Clearly, if the process of scattering is ignored we can write in the linear approximation for the change of the light intensity I:

$$\mathrm{dI} = -K_{\mathrm{ext}} \mathrm{I} \, \mathrm{d}l \, .$$

This underlines the experimental fact that the reduction of light intensity on the length $\mathrm{d}l$ is proportional to this length and the value of I itself. The coefficient of proportionality K_{ext} is called the extinction coefficient. Actually K_{ext} coincides with the absorption coefficient K_{abs} in this simple case. It follows that

$$\mathrm{I} = \mathrm{I}_0 \exp(-K_{\mathrm{ext}} l)$$

for a homogeneous ($K_{\mathrm{ext}} = \mathrm{const}$) layer, which is the well-known extinction law. Here I_0 is the incident light intensity at $l = 0$. This formula should be modified for light scattering media to account for light scattering from all other directions $\boldsymbol{\Omega}'$ to a given direction $\boldsymbol{\Omega}$. Then we have:

$$\mathrm{dI}(\mathbf{\Omega}) = -K_{\mathrm{ext}}\mathrm{I}(\mathbf{\Omega})\mathrm{d}l + \int_{4\pi} K_{\mathrm{sca}}(\mathbf{\Omega}, \mathbf{\Omega}')\mathrm{I}(\mathbf{\Omega})\,\mathrm{d}\mathbf{\Omega}'\,\mathrm{d}l\,,$$

where the differential scattering coefficient $K_{\mathrm{sca}}(\mathbf{\Omega}, \mathbf{\Omega}')$ describes the local scattering law. This formula can be written in the following form:

$$\frac{\mathrm{dI}(\mathbf{\Omega})}{\mathrm{d}l} = -K_{\mathrm{ext}}\mathrm{I}(\mathbf{\Omega}) + \int_{4\pi} K_{\mathrm{sca}}(\mathbf{\Omega}, \mathbf{\Omega}')\mathrm{I}(\mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega}'\,,$$

which is called the radiative transfer equation. The radiative transfer theory is concerned with the solution of this equation for scattering volumes (e.g., clouds), having different shapes, types of illuminations, and microstructure.

We will consider with solutions of RTE for a plane-parallel homogeneous turbid layer illuminated by a wide light beam. The interaction of solar radiation with extended cloud fields is well-approximated by the solution of this idealized problem. The geometry of the problem is given in Fig. 6.1. The wide light beam uniformly illuminates a plane-parallel scattering layer from above. We will assume that properties of the layer do not change in the horizontal direction. Then the light field changes only along the vertical coordinate Z (see Fig. 6.1). The intensity of light field also depends on the direction $\mathbf{\Omega}$, specified by the zenith angle ϑ and the azimuth φ. The main task of the radiative transfer theory is to calculate distributions $\mathrm{I}(\vartheta, \varphi, z)$. Usually only measurements of $\mathrm{I}(\vartheta, \varphi, 0)$ at the top of the turbid layer (reflected light) and $\mathrm{I}(\vartheta, \varphi, z_0)$ at the base of the turbid layer (transmitted light) are performed (see Fig. 6.1). Therefore, we will be concerned mostly with these two angular distributions.

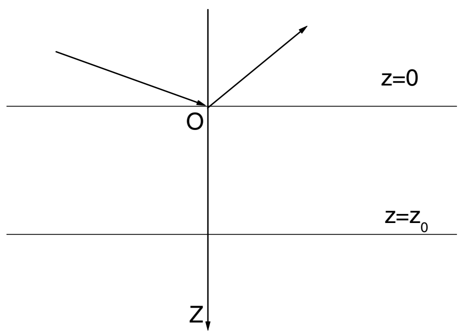

Fig. 6.1. The geometry of the problem

RTE for a plane-parallel light scattering vertically and horizontally homogeneous layer is reduced to the following simpler form:

$$\cos\vartheta\frac{\mathrm{dI}(\vartheta, \varphi)}{\mathrm{d}\tau} = -\mathrm{I}(\vartheta, \varphi) + \frac{\omega_0}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{\pi}\mathrm{d}\vartheta' p(\vartheta', \varphi' \to \vartheta, \varphi)\mathrm{I}(\vartheta', \varphi')\,,$$

if the polarization effects are ignored. Here we introduced the optical depth

$$\tau = \sigma_{\mathrm{ext}} z\,,$$

the phase function

$$p(\vartheta', \varphi' \to \vartheta, \varphi) = \frac{4\pi K_{\mathrm{sca}}(\vartheta', \varphi' \to \vartheta, \varphi)}{K_{\mathrm{sca}}}\,,$$

the scattering coefficient $K_{\mathrm{sca}} = K_{\mathrm{ext}} - K_{\mathrm{abs}}$, and the single scattering albedo

$$\omega_0 = \frac{K_{\mathrm{sca}}}{K_{\mathrm{ext}}}\,.$$

It is useful to make a separation of diffuse I and direct (or coherent) $I_c = A\delta(\cos\vartheta - \cos\vartheta_0)\delta(\varphi - \varphi_0)$ light in the general solution $\mathrm{I}(\vartheta, \varphi)$. The value of A is determined below and $\delta(x)$ is the delta function.

It is assumed that the layer is illuminated in the direction defined by the incidence zenith angle $\vartheta_0 = \arccos(\mu_0)$ and the azimuth φ_0. The density of the incident light flux on the area perpendicular to the beam is equal to F at the top of a scattering layer. The multiply scattered light is observed in the direction specified by the zenith observation angle $\vartheta = \arccos(\mu)$ and the azimuth φ. Namely, we write: $\mathrm{I}(\vartheta, \varphi) = I(\vartheta, \varphi) + I_c(\vartheta, \varphi)$. The substitution of this formula in RTE gives

$$\begin{aligned}\cos\vartheta \frac{\mathrm{d}I(\vartheta, \varphi)}{\mathrm{d}\tau} &= -I(\vartheta, \varphi) + \frac{\omega_0}{4\pi} \int_0^{2\pi} \mathrm{d}\varphi' \int_0^{\pi} \mathrm{d}\vartheta' p(\vartheta', \varphi' \to \vartheta, \varphi) I(\vartheta', \varphi') \\ &\quad + \frac{\omega_0}{4\pi} p(\vartheta_0, \varphi_0 \to \vartheta, \varphi) F \exp\left(-\frac{\tau}{\cos\vartheta_0}\right).\end{aligned}$$

The solution of this equation under boundary conditions stating that there is no diffuse light entering the turbid layer from above and below allows us to find $I(\vartheta, \varphi)$. $I_c(\vartheta, \varphi)$ is given simply by

$$I_c(\vartheta, \varphi) = F\delta(\cos\vartheta - \cos\vartheta_0)\delta(\varphi - \varphi_0) \exp\left(-\frac{\tau}{\cos\vartheta_0}\right).$$

The solution of RTE for the diffuse intensity I is simpler than that for the total intensity I because we avoid the necessity to deal with the divergence in the direction of incident light.

6.3 Reflection and transmission functions

Reflectance and transmittance of light by turbid layers is usually defined in terms of reflection R and transmission T functions. They relate incident light intensity $I_0(\vartheta_0, \varphi_0)$ with reflected $I_R(\mu, \varphi)$ and transmitted $I_T(\mu, \varphi)$ light intensity. Namely, it follows by definition:

$$
\begin{aligned}
I_R(\mu,\varphi) &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 R(\mu,\varphi,\mu',\varphi') I_0(\mu',\varphi')\mu'\,\mathrm{d}\mu' \,, \\
I_T(\mu,\varphi) &= \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 T(\mu,\varphi,\mu',\varphi') I_0(\mu',\varphi')\mu'\,\mathrm{d}\mu' \,.
\end{aligned}
$$

Reflection and transmission functions allow to find the intensity of reflected and transmitted light for arbitrary angular distributions of incident light with the intensity $I_0(\mu',\varphi')$.

If incident light is azimuthally independent, these formulas simplify:

$$
\begin{aligned}
I_R(\mu,\varphi) &= 2\int_0^1 \overline{R}(\mu,\varphi,\mu') I_0(\mu')\mu'\,\mathrm{d}\mu' \,, \\
I_T(\mu,\varphi) &= 2\int_0^1 \overline{T}(\mu,\varphi,\mu') I_0(\mu')\mu'\,\mathrm{d}\mu' \,,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{R}(\mu,\varphi,\mu') &= \frac{1}{2\pi}\int_0^{2\pi} R(\mu,\varphi,\mu',\varphi')\,\mathrm{d}\varphi' \,, \\
\overline{T}(\mu,\varphi,\mu') &= \frac{1}{2\pi}\int_0^{2\pi} T(\mu,\varphi,\mu',\varphi')\,\mathrm{d}\varphi' \,.
\end{aligned}
$$

The general equations given above can also be simplified for unidirectional illumination of a turbid layer by a wide beam (e.g., solar light). Then we can assume that

$$
I_0(\mu',\varphi') = F\delta(\mu'-\mu_0)\delta(\varphi'-\varphi_0) \,,
$$

where F is the incident light flux density at the top of a layer as introduced above and $\delta(x)$ is the delta function, having the following property:

$$
f(x_0) = \int_0^{\infty} \delta(x-x_0) f(x)\,\mathrm{d}x
$$

for arbitrary $f(x)$. Using this relation and equations for reflection and transmission functions given above, we arrive at the following results:

$$
\begin{aligned}
I_R(\mu,\varphi) &= \frac{F\mu_0 R(\mu,\varphi,\mu_0,\varphi_0)}{\pi} \,, \\
I_T(\mu,\varphi) &= \frac{F\mu_0 T(\mu,\varphi,\mu_0,\varphi_0)}{\pi} \,,
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
R(\mu,\varphi,\mu_0,\varphi_0) &= \frac{\pi I_R(\mu,\varphi)}{F\mu_0} \,, \\
T(\mu,\varphi,\mu_0,\varphi_0) &= \frac{\pi I_T(\mu,\varphi)}{F\mu_0} \,.
\end{aligned}
$$

These equations allow us to make the physical interpretation of reflection and transmission functions. Indeed, we have for an absolutely white Lambertian surface by definition:

$$\begin{aligned} P_R^L(\mu,\varphi) &= \int_{2\pi} I_R^L(\mu,\mu',\varphi,\varphi')\mu'\,\mathrm{d}\Omega' = \int_0^{2\pi}\mathrm{d}\varphi'\int_0^1 I_R^L(\mu,\mu',\varphi,\varphi')\mu'\,\mathrm{d}\mu' \\ &= \int_0^{2\pi}\mathrm{d}\varphi'\int_0^1 C\mu_0\mu'\,\mathrm{d}\mu' = \pi C\mu_0\,, \end{aligned}$$

where $P_R^L(\vartheta,\varphi)$ is the total power scattered by a unit area of a Lambertian surface into the upper hemisphere and we have used the fact that intensity of light reflected from a Lambertian surface is proportional to the cosine of the incidence angle $\mu_0(I_R^L = C\mu_0)$. The constant C can be found from the condition that the scattered $(P_R^L(\vartheta,\varphi))$ and incident (P_0) powers are equal in the case of the absolute white Lambertian surface by definition. We have for the incident power:

$$\begin{aligned} P_0 &= \int_{2\pi} I_0(\mu',\mu_0,\varphi',\varphi_0)\mu'\,\mathrm{d}\Omega' \\ &= F\int_0^{2\pi}\mathrm{d}\varphi'\int_0^1 \delta(\mu'-\mu_0)\delta(\varphi'-\varphi_0)\mu'\,\mathrm{d}\mu' = F\mu_0 \end{aligned}$$

and, therefore: $C = F/\pi$. It means that intensity of light reflected from an absolutely Lambertian surface is given by:

$$I_R^L(\vartheta_0,\vartheta,\varphi) = \frac{F}{\pi}\mu_0\,.$$

We conclude that $R(\mu,\vartheta,\mu_0,\vartheta_0)$ is equal to the ratio of light reflected from a given surface I_R to the value of I_R^L:

$$R = I_R/I_R^L\,.$$

It means that $R \equiv 1$ for a Lambertian ideally white surface.

Accordingly, it follows that

$$T = I_T/I_R^L\,.$$

The results of calculations will be mostly presented in terms of functions R and T in this work. These functions do not depend on the intensity of incident light, and characterize inherent properties of a turbid layer. The integration of reflection and transmission functions with respect to angles allows us to find the cloud plane r_d and spherical r albedos, the diffuse t_d and global t transmittances, the absorptance a_d and the global absorptance a as specified in Table 6.1.

Table 6.1. Radiative transfer characteristics ($\overline{R}$ and $\overline{T}$ are azimuthally averaged reflection and transmission functions, respectively)

Radiative characteristic	Symbol	Definition
Plane albedo	$r_d(\mu_0)$	$2\int_0^1 \overline{R}(\mu_0,\mu)\mu\,\mathrm{d}\mu$
Spherical albedo	r	$2\int_0^1 r_d(\mu_0)\mu_0\,\mathrm{d}\mu_0$
Diffuse transmittance	$t_d(\mu_0)$	$2\int_0^1 \overline{T}(\mu_0,\mu)\mu\,\mathrm{d}\mu$
Global transmittance	t	$2\int_0^1 t_d(\mu_0)\mu_0\,\mathrm{d}\mu_0$
Absorptance	$a_d(\mu_0)$	$1-r_d(\mu_0)-t_d(\mu_0)$
Global absorptance	a	$1-r-t$

6.4 Asymptotic theory

6.4.1 Auxiliary functions and relationships

Let us find the solution of RTE valid for optically thick turbid media ($z_0 \gg K_{\mathrm{ext}}^{-1}$, see Fig. 6.1). It is known that light intensity in deep layers of optically thick light scattering media is azimuthally independent. Then the radiative transfer equation can be written in the following form:

$$\eta\frac{\mathrm{d}I(\tau,\eta)}{\mathrm{d}\tau} = -I(\tau,\eta)+B(\tau,\eta)+B_0(\tau,\eta)\,,$$

where

$$\begin{aligned}B(\tau,\eta) &= \frac{\omega_0}{2}\int_{-1}^{1} p(\eta,\eta')I(\tau,\eta')\,\mathrm{d}\eta'\,,\\ B_0(\tau,\eta) &= \frac{\omega_0 F}{4\pi}p(\eta,\xi)e^{-\tau/\xi}\,,\end{aligned}$$

$\xi=\cos\vartheta_0$, $\eta=\cos\vartheta$, and

$$p(\eta,\xi)=\frac{1}{2\pi}\int_0^{2\pi} p(\eta,\xi,\varphi)\,\mathrm{d}\varphi$$

is the azimuthally averaged phase function. This result can be obtained from RTE (see section 6.2) performing integration with respect to the azimuth.

Let us assume that $\tau\to\infty$. Then it follows that $B_0(\tau,\eta)\to 0$ and (Sobolev, 1975)

$$I(\tau,\eta)=i(\eta)e^{-k\tau}\,.$$

The last equation corresponds to the so-called deep-layer regime, when parameters η and τ are decoupled. Then the light field intensity decreases with the distance from the illuminated boundary preserving the scattered light angular pattern given by the function $i(\eta)$. The value of I decreases in e times at the optical depth $\tau_e = 1/k$. Both the function $i(\eta)$ and the diffusion exponent k play an important role in the theory considered here. It is interesting that these characteristics of the deep-layer regime also define the intensity of transmitted and reflected light of optically thick layers as will be shown below.

It is easy to derive, using equations given above, that

$$(1 - k\eta)i(\eta) = \frac{\omega_0}{2} \int_{-1}^{1} p(\eta, \eta')i(\eta')\,\mathrm{d}\eta' ,$$

which is called the deep regime radiative transfer equation (DRTE). This integral equation is usually solved numerically. Let us assume that $p = 1$. Then we have:

$$i(\eta) = \frac{\omega_0}{2(1 - k\eta)} \int_{-1}^{1} i(\eta')\,\mathrm{d}\eta'$$

or

$$i(\eta) = \frac{D}{1 - k\eta} ,$$

where

$$D = \frac{\omega_0}{2} \int_{-1}^{1} i(\eta')\,\mathrm{d}\eta'$$

does not depend on the angle. Note that $i(\eta)$ satisfies DRTE for any constants D and, therefore,

$$i(\eta) = \frac{1}{1 - k\eta} ,$$

where we used the normalization condition: $D = 1$. The diffusion constant k can be found substituting the last equation in DRTE. Then we have:

$$\frac{\omega_0}{2k} \ln\left(\frac{1 + k}{1 - k}\right) = 1$$

at $p = 1$. This equation allows to find k at arbitrary ω_0 and $p = 1$.

We can also write:

$$(1 + k\eta)i(-\eta) = \frac{\omega_0}{2} \int_{-1}^{1} p(-\eta, \eta')i(\eta')\,\mathrm{d}\eta'$$

or

$$(1 + k\eta)i(-\eta) = \frac{\omega_0}{2} \int_{-1}^{1} p(\eta, \eta')i(-\eta')\,\mathrm{d}\eta' ,$$

where we used the property: $p(-\eta, -\eta') = p(\eta, \eta')$.

Let us establish now the relationship between the intensity $i^{\downarrow}(\eta)$ for light propagated downwards and the intensity $i^{\uparrow}(-\eta)$ for light propagated upwards.

Arrows and signs before η indicate the direction of light propagation. For this we consider a cut parallel to the upper boundary but at a large optical depth. The correspondent plane at $\tau \gg 1$ is illuminated not only by light coming from above and having the intensity i_a but also by light coming from below and reflected from the layer above the plane of cut. We denote this contribution to the total intensity as i_b. Then we have:

$$i^{\downarrow}(\eta) = i_a(\eta) + i_b(\eta)\,.$$

So the function $i^{\downarrow}(\eta)$ can be presented as a sum of two terms. Clearly, $i_a(\eta)$ is proportional to the angular distribution $u(\eta)$ of light transmitted by the upper layer:

$$i_a(\eta) = Mu(\eta)\,,$$

where M is the unknown proportionality constant. We will find this constant at later stages of our derivations. Also it follows for the intensity $i_b(\eta)$ by definition (see section 6.3) that

$$i_b(\eta) = 2\int_0^1 R(\eta,\eta')i(-\eta')\eta'\,\mathrm{d}\eta'\,,$$

where $R(\eta,\eta')$ is the azimuthally averaged reflection function of the upper layer under illumination from below ($\eta > 0, \eta' > 0$). This layer can be chosen to be arbitrarily thick. So we will assume that $R(\eta,\eta')$ coincides with the azimuthally averaged reflection function of a semi-infinite layer $R_\infty(\eta,\eta')$.

Summing up, it follows that

$$i^{\downarrow}(\eta) = Mu(\eta) + 2\int_0^1 R_\infty(\eta,\eta')i(-\eta')\eta'\,\mathrm{d}\eta'\,.$$

Let us find M. We multiply the last equation by $\eta i^{\downarrow}(\eta)$ and integrate it from 0 to 1 with respect to η. Then we have:

$$\int_0^1 i^{\downarrow^2}(\eta)\eta\,\mathrm{d}\eta = M\int_0^1 u(\eta)i^{\downarrow}(\eta)\eta\,\mathrm{d}\eta + \mathfrak{I}$$

where the two-dimensional integral

$$\mathfrak{I} = 2\int_0^1 i^{\downarrow}(\eta)\eta\,\mathrm{d}\eta\int_0^1 i^{\uparrow}(-\eta')R_\infty(\eta,\eta')\eta'\,\mathrm{d}\eta'$$

can be simplified. For this we note that it follows by definition (see section 6.3) that

$$i^{\uparrow}(-\eta') = 2\int_0^1 i_{\downarrow}(\eta)R_\infty(\eta,\eta')\eta\,\mathrm{d}\eta$$

and, therefore,

$$\mathfrak{I} = \int_0^1 i^{\uparrow^2}(-\eta')\eta'\,\mathrm{d}\eta'$$

or

$$\Im = -\int_{-1}^{0} i^{\uparrow^2}(\eta')\eta' \, \mathrm{d}\eta' \, .$$

Therefore, it follows, omitting arrows, that

$$M = \mathbb{C} \int_{-1}^{1} i^2(\eta)\eta \, \mathrm{d}\eta \, ,$$

where

$$\mathbb{C} = \left[\int_{0}^{1} u(\eta) i(\eta) \eta \, \mathrm{d}\eta \right]^{-1} \, .$$

We will use the normalization condition: $\mathbb{C} = 2$. Then it follows:

$$M = 2 \int_{-1}^{1} i^2(\eta)\eta \, \mathrm{d}\eta \, .$$

We present the equation for M together with other important relations in Table 6.2. The constant N defined in the property 6.8 (see Table 6.2) will be used in further derivations devoted to studies of relationships between auxiliary functions

$$P(\tau) = \int_{-1}^{1} \eta i(\eta) I(\tau, \eta) \, \mathrm{d}\eta$$

Table 6.2. Main equations and constants

No.	Property
6.1	$(1 - k\eta) i(\eta) = \frac{\omega_0}{2} \int_{-1}^{1} p(\eta, \eta') i(\eta') \, \mathrm{d}\eta'$
6.2	$(1 + k\eta) i(-\eta) = \frac{\omega_0}{2} \int_{-1}^{1} p(\eta, \eta') i(-\eta') \, \mathrm{d}\eta'$
6.3	$\frac{\omega_0}{2} \int_{-1}^{1} i(\eta) \, \mathrm{d}\eta = 1$
6.4	$i(-\eta) = 2 \int_{0}^{1} i(\xi) R_\infty(\xi, \eta) \xi \, \mathrm{d}\xi$
6.5	$i(\eta) = 2 \int_{0}^{1} i(-\xi) R_\infty(\xi, \eta) \xi \, \mathrm{d}\xi + M u(\eta)$
6.6	$2 \int_{0}^{1} u(\eta) i(\eta) \eta \, \mathrm{d}\eta = 1$
6.7	$M = 2 \int_{-1}^{1} i^2(\eta) \eta \, \mathrm{d}\eta$
6.8	$N = 2 \int_{0}^{1} i(-\eta) u(\eta) \eta \, \mathrm{d}\eta$
6.9	$\eta \frac{\mathrm{d}I}{\mathrm{d}\tau} = -I + B + B_0$

and

$$Q(\tau) = \int_{-1}^{1} \eta i(-\eta) I(\tau, \eta)\, \mathrm{d}\eta\,.$$

The relationships between functions $P(\tau)$ and $Q(\tau)$ are of importance for the derivation of asymptotical equations for reflection and transmission functions valid as the optical thickness $\tau_0 = K_{\mathrm{ext}} z_0 \to \infty$. Let us show this.

First of all, we note that it follows after multiplication of eq. (6.9) in Table 6.2 by $i(\eta)$ and integration from -1 to 1:

$$\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = -kP(\tau) + P_0(\tau)\,,$$

where

$$P_0(\tau) = \int_{-1}^{1} i(\eta) B_0(\tau, \eta)\, \mathrm{d}\eta$$

and we used the equality

$$-kP(\tau) = \int_{-1}^{1} B(\tau, \eta) i(\eta)\, \mathrm{d}\eta - \int_{-1}^{1} i(\eta) I(\tau, \eta)\, \mathrm{d}\eta\,.$$

This equality can be obtained from eq. (6.1) in Table 6.2. Let us show it. We have after multiplying eq. (6.1) in Table 6.2 by $I(\tau, \eta)$ and integrating this equation from -1 to 1 with respect to η:

$$\int_{-1}^{1} I(\tau, \eta) i(\eta)\, \mathrm{d}\eta - kP(\tau) = \frac{\omega_0}{2} \int_{-1}^{1} \mathrm{d}\eta \int_{-1}^{1} I(\tau, \eta) p(\eta, \eta') i(\eta')\, \mathrm{d}\eta'$$

or

$$\int_{-1}^{1} I(\tau, \eta) i(\eta)\, \mathrm{d}\eta - kP(\tau) = \int_{-1}^{1} B(\tau, \eta) i(\eta)\, \mathrm{d}\eta\,,$$

where we used the property: $p(\eta, \eta') = p(\eta', \eta)$. This completes the proof.

The next step is to find $P(\tau)$ from the differential equation given above. For this we use the following substitution:

$$P(\tau) = f(\tau) e^{-k\tau}\,.$$

Then it follows that

$$\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} = P_0(\tau) e^{k\tau}$$

or

$$f_{\tau_1}^{\tau} = \int_{\tau_1}^{\tau} P_0(t) e^{kt}\, \mathrm{d}t\,.$$

It means that

$$f(\tau) = f(\tau_1) + \int_{\tau_1}^{\tau} P_0(t) e^{kt}\, \mathrm{d}t\,.$$

So we have:

$$P(\tau) = f(\tau_1)e^{-k\tau} + e^{-k\tau}\int_{\tau_1}^{\tau} P_0(t)e^{kt}\,\mathrm{d}t\,.$$

The value of τ_1 can be found from boundary conditions. In particular, we interested in the diffuse light. Diffused light does not enter the medium from above or below of a turbid slab ($I(0,\eta) = 0$ for $\eta > 0$ and $I(\tau_0,\eta) = 0$ for $\eta < 0$). Therefore, we have: $\tau_1 = 0$. Then the boundary condition at the upper boundary is satisfied. Finally, it follows that

$$P(\tau) = P(0)e^{-k\tau} + \int_0^{\tau} P_0(t)e^{k(t-\tau)}\,\mathrm{d}t\,.$$

A similar relationship can be obtained for $Q(\tau,\eta)$. Namely, we have after multiplication of eq. (6.9) in Table 6.2 by $i(-\eta)$ and performing the integration from -1 to 1:

$$\frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau} = kQ(\tau) + Q_0(\tau)\,,$$

where

$$Q_0(\tau) = \int_{-1}^{1} i(-\eta)B_0(\tau,\eta)\,\mathrm{d}\eta\,.$$

This equation differs from the corresponding equation for $P(\tau)$ only in the sign before k. So it follows that

$$Q(\tau) = \psi(\tau_1^*)e^{k\tau} + e^{k\tau}\int_{\tau_1^*}^{\tau} Q_0(t)e^{-kt}\,\mathrm{d}t\,,$$

where it was assumed that

$$Q(\tau) = \psi(\tau)e^{k\tau}\,.$$

The value of τ_1^* can be found from the boundary condition at the lower boundary of a medium. Namely, we have: $\tau_1^* = \tau_0$. Therefore, it follows that

$$Q(\tau) = Q(\tau_0)e^{k(\tau-\tau_0)} + \int_{\tau_0}^{\tau} Q_0(t)e^{-k(t-\tau)}\,\mathrm{d}t\,.$$

This equation gives an identity at $\tau = \tau_0$ due to the accurate account for the boundary conditions.

Summing up, we have the following important relationships:

$$\begin{aligned} P(\tau) &= P(0)e^{-k\tau} + V(\tau)\,,\\ Q(\tau) &= Q(\tau_0)e^{-k(\tau-\tau_0)} + W(\tau)\,, \end{aligned}$$

where

$$\begin{aligned} V(\tau) &= \int_0^{\tau} P_0(t)e^{k(t-\tau)}\,\mathrm{d}t\,,\\ W(\tau) &= \int_{\tau_0}^{\tau} Q_0(t)e^{-k(t-\tau)}\,\mathrm{d}t\,. \end{aligned}$$

These fundamental relationships are valid for any τ and for any light sources represented by B_0. They can be used for the derivation of a number of important results of light scattering media optics.

We will use a particular case at $\tau = \tau_0$ in the first equation and a case $\tau = 0$ in the second equation. Then it follows that

$$\begin{aligned} P(\tau_0) &= P(0)\exp(-k\tau_0) + V(\tau_0)\,, \\ Q(0) &= Q(\tau_0)\exp(-k\tau_0) + W(0)\,, \end{aligned}$$

where

$$\begin{aligned} V(\tau_0) &= e^{-k\tau_0}\int_0^{\tau_0} \mathrm{d}t \int_{-1}^{1} i(\eta)\frac{\omega_0 F}{4\pi} p(\eta,\xi) e^{-t\left(\frac{1}{\xi}-k\right)}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\left(e^{-k\tau_0} - e^{-\frac{\tau_0}{\xi}}\right)\xi i(\xi) F\,, \end{aligned}$$

$$\begin{aligned} W(0) &= \int_{\tau_0}^{0} e^{-kt}\,\mathrm{d}t \int_{-1}^{1} i(-\eta)\frac{\omega_0 F}{4\pi} p(\eta,\xi) e^{-t/\xi}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\left(e^{-\left(k+\frac{1}{\xi}\right)\tau_0} - 1\right)\xi i(-\xi) F\,, \end{aligned}$$

where we used properties 1 and 2 in Table 6.2. Therefore, neglecting small numbers proportional to $e^{-\tau_0/\xi}$, it follows that

$$\begin{aligned} P(0) &= P(\tau_0)e^{k\tau_0} - \frac{\xi i(\xi) F}{2\pi}\,, \\ Q(0) &= Q(\tau_0)e^{-k\tau_0} - \frac{\xi i(-\xi) F}{2\pi}\,. \end{aligned}$$

These are auxiliary relations we were bound to establish from the very start. They can be also written in the following form:

$$\begin{aligned} i(\xi) &= \frac{2\pi e^{k\tau_0}}{\xi F}\int_{-1}^{1} I(\eta,\tau_0) i(\eta)\eta\,\mathrm{d}\eta - \frac{2\pi}{\xi F}\int_{-1}^{1} I(\eta,0) i(-\eta)\eta\,\mathrm{d}\eta\,, \\ i(-\xi) &= \frac{2\pi e^{-k\tau_0}}{\xi F}\int_{-1}^{1} I(\eta,\tau_0) i(-\eta)\eta\,\mathrm{d}\eta - \frac{2\pi}{\xi F}\int_{-1}^{1} I(\eta,0) i(-\eta)\eta\,\mathrm{d}\eta\,. \end{aligned}$$

Now we take into account that

$$I(-\eta,0) = \frac{\xi F}{\pi}\overline{R}(\xi,\eta)$$

at $\eta > 0$ and $I(-\eta,0) = 0$, otherwise. Also it follows that

$$I(\eta,\tau_0) = \frac{\xi F}{\pi}\overline{T}(\xi,\eta)$$

at $\eta > 0$ and $I(\eta,\tau_0) = 0$, otherwise. This means that we can write:

$$-\int_{-1}^{1} I(\eta,0)i(\eta)\eta\,\mathrm{d}\eta = -\int_{1}^{-1} I(-\eta,0)i(-\eta)\eta\,\mathrm{d}\eta = -\int_{1}^{0} I(-\eta,0)i(-\eta)\eta\,\mathrm{d}\eta$$

and

$$-\int_{-1}^{1} I(\eta,0)i(-\eta)\eta\,\mathrm{d}\eta = -\int_{1}^{-1} I(-\eta,0)i(\eta)\eta\,\mathrm{d}\eta = -\int_{1}^{0} I(-\eta,0)i(\eta)\eta\,\mathrm{d}\eta\,.$$

Similar relationships can be written for integrals containing $I(\eta,\tau_0)$. Then one obtains:

$$\begin{aligned} i(\xi) &= 2e^{k\tau_0}\int_0^1 \overline{T}(\eta,\xi,\tau_0)i(\eta)\eta\,\mathrm{d}\eta + 2\int_0^1 \overline{R}(\eta,\xi,\tau_0)i(-\eta)\,\mathrm{d}\eta\,, \\ i(-\xi) &= 2e^{-k\tau_0}\int_0^1 \overline{T}(\eta,\xi,\tau_0)i(-\eta)\eta\,\mathrm{d}\eta + 2\int_0^1 \overline{R}(\eta,\xi,\tau_0)i(\eta)\eta\,\mathrm{d}\eta\,. \end{aligned}$$

6.4.2 Asymptotic equations

The general form of functions $\overline{R}(\eta,\xi,\tau_0)$ and $\overline{T}(\eta,\xi,\tau_0)$ can be obtained using physical arguments. In particular, $\overline{T}$ should be proportional to $u(\eta)$ (and, actually, due to the reciprocity principle also to $u(\xi)$). Therefore, we have:

$$T(\eta,\xi,\tau_0) = \alpha(\tau_0)u(\eta)u(\xi)\,,$$

where $\alpha(\tau_0)$ is the unknown function.

Let us consider now a semi-infinite layer and take a cut at a large optical thickness τ_0. Then we can represent $R_\infty(\eta,\xi)$ as a sum of reflection from upper layer $\overline{R}(\eta,\xi,\tau_0)$ and light transmitted by the upper layer and reflected back. The angular distribution of the transmitted light should be proportional to $u(\eta)u(\xi)$ as was specified above. So we have:

$$R_\infty(\eta,\xi) = \overline{R}(\tau,\eta,\xi) + \beta(\tau_0)u(\eta)u(\xi)\,.$$

Let us find unknown functions $\alpha(\tau_0)$ and $\beta(\tau_0)$ using expressions for $i(\pm\xi)$ derived above and also properties specified in Table 6.2. Then it follows that

$$\begin{aligned} i(\xi) &= 2e^{k\tau_0}\int_0^1 \alpha(\tau_0)u(\eta)u(\xi)i(\eta)\eta\,\mathrm{d}\eta \\ &\quad +2\int_0^1 \left(R_\infty(\eta,\xi) - \beta(\tau_0)u(\eta)u(\xi)\right)i(-\eta)\eta\,\mathrm{d}\eta\,, \end{aligned}$$

$$\begin{aligned} i(-\xi) &= 2e^{-k\tau_0}\int_0^1 \alpha(\tau_0)u(\eta)u(\xi)i(-\eta)\eta\,\mathrm{d}\eta \\ &\quad +2\int_0^1 \left(R_\infty(\eta,\xi) - \beta(\tau_0)u(\eta)u(\xi)\right)i(\eta)\,\mathrm{d}\eta\,. \end{aligned}$$

One obtains, using properties given in Table 6.2:

$$
\begin{aligned}
i(\xi) &= e^{k\tau_0}\alpha(\tau_0)u(\xi) + i(\xi) - Mu(\xi) - \beta(\tau_0)u(\xi)N\,, \\
i(-\xi) &= i(-\xi) - \beta(\tau_0)u(\xi) + \alpha(\tau_0)Ne^{-k\tau_0}u(\xi)\,,
\end{aligned}
$$

where we introduced the integral

$$N = 2\int_0^1 u(\eta)i(-\eta)\eta\,\mathrm{d}\eta\,.$$

Therefore, it follows that

$$
\begin{gathered}
\alpha(\tau_0) - Me^{-k\tau_0} - \beta Ne^{-k\tau_0} = 0\,, \\
\beta(\tau_0) = \alpha(\tau_0)Ne^{-k\tau_0}\,.
\end{gathered}
$$

So one obtains:

$$\alpha(\tau_0) = \frac{Me^{-k\tau_0}}{1 - N^2e^{-2k\tau_0}}\,, \quad \beta(\tau_0) = \frac{MNe^{-2k\tau_0}}{1 - N^2e^{-2k\tau_0}}$$

Finally, we have, as $\tau_0 \to \infty$ at arbitrary ω_0 and $p(\theta)$:

$$
\begin{aligned}
\overline{R}(\xi,\eta) &= R_\infty(\xi,\eta) - \overline{T}(\xi,\eta)Ne^{-k\tau_0}\,, \\
\overline{T}(\xi,\eta) &= \frac{Me^{-k\tau_0}}{1 - N^2e^{-2k\tau_0}}u(\eta)u(\xi)\,.
\end{aligned}
$$

These formulas are central equations of the light scattering media optics. The importance of these equations is due to the fact that the dependence of radiative characteristics on τ_0 is given explicitly. Our next task is to derive approximate equations for constants k, M, N and functions $u(\eta)$, $R_\infty(\xi,\eta)$ in a number of particular cases.

The dependence of the transmitted light on the azimuth is weak. So we may write:

$$
\begin{aligned}
R(\xi,\eta,\varphi) &= R_\infty(\xi,\eta,\varphi) - T(\xi,\eta)Ne^{-k\tau_0}\,, \\
T(\xi,\eta) &= \frac{Me^{-k\tau_0}}{1 - N^2e^{-2k\tau_0}}u(\eta)u(\xi)\,,
\end{aligned}
$$

where we have accounted for the fact that the reflection function of a semi-infinite turbid medium does depend on the azimuth.

The choice of the normalization condition for the function $i(\eta)$ and also for the function $u(\eta)$ is arbitrary. We have followed the notation of Sobolev (1975). It differs from that in the corresponding equations used by van de Hulst (1980). For instance, van de Hulst's diffusion pattern $P(\eta)$ must be divided by ω_0 to yield $i(\eta)$. His escape function $K(\eta)$ must be multiplied by ω_0 to yield $u(\eta)$, and his M equals that used by Sobolev multiplied by ω_0^2. These differences do not lead to extra factors in main equations given in this section. They are also of no importance at $\omega_0 = 1$.

6.4.3 Weak absorption

Equations given above can be simplified considerably for the case of values of ω_0 close to unity. Therefore, we need to find approximate expressions for functions $R_\infty(\xi,\eta)$, $u(\eta)$ and also for parameters k, M, N as $\omega_0 \to 1$. Let us concentrate on this problem now.

6.4.3.1 The constants k, M and the diffuse light field in deep layers

The parameter M depends on the diffuse light intensity $i(\eta)$ in deep layers of a turbid medium:

$$M = 2\int_{-1}^{1} i^2(\eta)\eta\,\mathrm{d}\eta\,.$$

So we need to study functions $i(\eta)$ as $\omega_0 \to 1$. The radiative transfer equation for the normalized light intensity $i(\eta)$ deep inside of a turbid medium has the following form:

$$(1-k\eta)i(\eta) = \frac{\omega_0}{2}\int_{-1}^{1} p(\eta,\eta')i(\eta')\,\mathrm{d}\eta'\,,$$

where $p(\eta,\eta')$ is the azimuthally averaged phase function, ω_0 is the single scattering albedo and k is the diffusion exponent. The normalization condition for $i(\eta)$ has the following form:

$$\frac{\omega_0}{2}\int_{-1}^{1} i(\eta')\,\mathrm{d}\eta' = 1\,.$$

We use the following expansions:

$$p(\eta,\eta') = \sum_{n=0}^{\infty} x_n P_n(\eta)P_n(\eta')$$

and

$$i(\eta) = \sum_{n=0}^{\infty} \sigma_n P_n(\eta)\,.$$

The task is to find σ_n from the set of x_n. Substituting these expressions in DRTE, we have:

$$B = \sum_{n=0}^{\infty} \sigma_n P_n(\eta) - k\eta\sum_{n=0}^{\infty} \sigma_n P_n(\eta)\,,$$

where

$$B = \frac{\omega_0}{2}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} x_l\sigma_n P_l(\eta)P_l(\eta')P_n(\eta')\,\mathrm{d}\eta'$$

or

$$B = \omega_0\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \sigma_n x_l \delta_{nl}[2n+1]^{-1}P_n(\eta)$$

and after simplifications:

$$B = \omega_0\sum_{n=0}^{\infty} x_n\sigma_n[2n+1]^{-1}P_n(\eta)\,,$$

where we used the fact that Legendre polynomials are orthogonal. This means that

$$\int_{-1}^{1} P_n(\eta) P_l(\eta)\,\mathrm{d}\eta = 2\delta_{nl}[2n+1]^{-1}\,,$$

where δ_{nl} is the Kronecker symbol.

Therefore, it follows that

$$\frac{1}{k}\sum_{n=0}^{\infty}\sigma_n\left\{1-\frac{x_n\omega_0}{2n+1}\right\}P_n(\eta)=\sum_{n=0}^{\infty}\sigma_n\left\{\frac{n+1}{2n+1}P_{n+1}(\eta)+\frac{n}{2n+1}P_{n-1}(\eta)\right\}\,,$$

where we have used the property:

$$\eta P_n(\eta)=\frac{n+1}{2n+1}P_{n+1}(\eta)+\frac{n}{2n+1}P_{n-1}(\eta)\,.$$

The expressions for

$$\zeta(\eta)=\sum_{n=0}^{\infty}\sigma_n\frac{n+1}{2n+1}P_{n+1}(\eta)$$

and

$$\upsilon(\eta)=\sum_{n=0}^{\infty}\sigma_n\frac{n}{2n+1}P_{n-1}(\eta)$$

can be written as:

$$\zeta(\eta)=\sum_{l=0}^{\infty}\sigma_{l-1}\frac{l}{2l-1}P_l(\eta)$$

and

$$\upsilon(\eta)=\sum_{s=0}^{\infty}\sigma_{s+1}\frac{s+1}{2s+3}P_l(\eta)\,,$$

where $l=n+1$, $s=n-1$.

Therefore, we have:

$$\sum_{m=0}^{\infty}\left[\frac{1}{k}\sigma_m-\frac{x_m\omega_0}{(2m+1)k}\sigma_m-\frac{m}{2m-1}\sigma_{m-1}-\frac{m+1}{2m+3}\sigma_{m+1}\right]P_m(\eta)=0$$

at arbitrary η. This means that

$$\frac{1}{k}\sigma_m-\frac{x_m\omega_0}{(2m+1)k}\sigma_m-\frac{m}{2m-1}\sigma_{m-1}-\frac{m+1}{2m+3}\sigma_{m+1}=0$$

or

$$\sigma_{m+1}=\frac{(2m+3)(2m-\omega_0 x_m+1)}{(2m+1)(m+1)k}\sigma_m-\frac{(2m+3)m}{(2m-1)(m+1)}\sigma_{m-1}\,.$$

We have at $m=0$:

$$\sigma_1=\frac{3\sigma_0(1-\omega_0)}{k}\,.$$

Let us derive the expression for the value of σ_0 now. It follows that

$$\sigma_m = \frac{2m+1}{2} \int_{-1}^{1} i(\eta) P_m(\eta)\,\mathrm{d}\eta$$

and, therefore,

$$\sigma_0 = \frac{1}{2} \int_{-1}^{1} i(\eta)\,\mathrm{d}\eta\,.$$

Comparing this result with property 6.3 in Table 6.2, we derive: $\sigma_0 = \omega_0^{-1}$, and, therefore,

$$\sigma_1 = \frac{3(1-\omega_0)}{k\omega_0}\,.$$

This allows us to obtain the following expansion for $i(\eta)$ as $\omega_0 \to 1$:

$$i(\eta) = \omega_0^{-1} \left\{ 1 + 3k^{-1}(1-\omega_0)\eta \right\}\,,$$

where we neglected higher-order terms with respect to the probability of photon absorption $\beta \equiv 1-\omega_0$. So it follows, as $\omega_0 \to 1$, that

$$i(\eta) = 1 + 3k^{-1}(1-\omega_0)\eta\,.$$

Recurrence relations allow us to find σ_m and $i(\eta)$ at any k. We will not consider this problem here, however, but rather concentrate on the derivation of the approximate equation for k valid as $\omega_0 \to 1$.

For this we introduce:

$$\Upsilon_m = \frac{\sigma_m}{\sigma_{m-1}}\,.$$

Then it follows that

$$\Upsilon_{m+1} = \frac{(2m+3)(2m-\omega_0 x_m+1)}{(2m+1)(m+1)k} - \frac{(2m+3)m}{(2m-1)(m+1)\Upsilon_m}$$

and

$$\Upsilon_m = \frac{(2m+3)m}{(2m-1)(m+1)\left[\dfrac{(2m+3)(2m-\omega_0 x_m+1)}{(2m+1)(m+1)k} - \Upsilon_{m+1}\right]}$$

or

$$\Upsilon_m = \frac{(2m+3)(2m+1)mk}{(2m+3)(2m-1)(2m+1-\omega_0 x_m) - \varepsilon_m}\,,$$

where

$$\varepsilon_m = (4m^2-1)(1+m)k\Upsilon_{m+1}\,.$$

We are interested in the asymptotic solution valid as $k \to 0$. So we can ignore ε_m and derive at $m=1$:

$$\Upsilon_1 = \frac{3k}{(3-\omega_0 x_1)}\,.$$

Therefore, it follows that

$$\sigma_1 = \frac{3k}{(3-\omega_0 x_1)}\sigma_0$$

or

$$\sigma_1 = \frac{3k}{(3-\omega_0 x_1)\omega_0} .$$

This formula must produce the same result as the expression for σ_1 derived above. It means that

$$k = \sqrt{3(1-\omega_0 g)(1-\omega_0)} ,$$

where $g = x_1/3$ is the asymmetry parameter. This important equation shows that the intensity in the deep layers of optically thick weakly absorbing media clouds decreases more quickly for smaller values of the asymmetry parameter g (less extended in the forward direction phase functions). Our derivations are valid as $\omega_0 \to 1$ only. So we can also write: $k = \sqrt{3(1-\omega_0)(1-g)}$. The approximate expression for $i(\eta)$ given above can be written using the similarity parameter

$$s = \sqrt{\frac{1-\omega_0}{1-\omega_0 g}} .$$

Namely, we have:

$$i(\eta) = 1 + \frac{s\eta}{\sqrt{3}} .$$

Note that it follows as $\omega_0 \to 1$: $s \approx \sqrt{\dfrac{1-\omega_0}{1-g}}$.

The angular pattern $i(\eta)$ does not depend on the choice of a particular light scattering medium if s kept constant. The function $i(\eta)$ is completely determined by the similarity parameter s as $\omega_0 \to 1$. Therefore, light scattering media having different values of ω_0 and g but the same s have very similar light angular distributions in the deep-layer regime.

The parameters k and s are of a crucial importance for the theory considered here. We must expect that constants and functions in asymptotic equations must depend on these values. In particular, taking into account property 6.7 in Table 6.2, we obtain:

$$M = \frac{8s}{\sqrt{3}}$$

as $k \to 0$.

6.4.3.2 The constant *N* and the escape function

The expansion of $u(\eta)$ with respect to the diffusion coefficient k can be presented as

$$u(\eta) = \sum_{n=0}^{\infty} k^n u_n(\eta) .$$

We are interested only in the case of weak absorption. Then it follows that

$$u(\eta) = u_0(\eta) + k u_1(\eta) .$$

The task is to calculate the function $u_1(\eta)$. This will be performed in two steps. First of all we note that the weak absorption of light does not alter the single scattering law considerably. The angular distribution of emerging light $u(\eta)$ is determined largely by the scattering processes. So it is safe to assume that $u(\eta) \approx u_0(\eta)$ as $k \to 0$ or $u_1(\eta) = bu_0(\eta)$, where the constant b should be determined. Clearly, due to physical reasons we should have: $u(\eta) < u_0(\eta)$ and $b < 0$. Therefore, absorption plays the role of a veil in this case. It reduces the contrast but it does not change details of the scattering pattern. We start from the expression:

$$2\int_0^1 u(\eta)i(\eta)\eta\,\mathrm{d}\eta = 1\,.$$

Let us substitute the following expansions in this formula:

$$u(\eta) = u_0(\eta)(1 + bk)$$

and

$$i(\eta) = 1 + ak\eta\,,$$

where we assume that $\omega_0 \to 1$ and, therefore,

$$k = \sqrt{3(1-g)(1-\omega_0)}\,, \quad s = \sqrt{\frac{1-\omega_0}{1-g}}\,, \quad \text{and} \quad a = (1-g)^{-1}\,.$$

Then it follows that

$$2\int_0^1 u_0(\eta)\eta\,\mathrm{d}\eta + 2bk\int_0^1 u_0(\eta)\eta\,\mathrm{d}\eta + 2ak\int_0^1 u_0(\eta)\eta^2\,\mathrm{d}\eta = 1\,,$$

where we neglected high-order terms. So we have:

$$b = -2a\int_0^1 u_0(\eta)\eta^2\,\mathrm{d}\eta\,,$$

where we have accounted for the fact that (see property 6.6 in Table 6.2 at $\omega_0 = 1$, $i \equiv 1$)

$$2\int_0^1 u_0(\eta)\eta\,\mathrm{d}\eta = 1\,.$$

Finally, it follows that

$$b = -\frac{2\nu}{1-g}\,,$$

where we have accounted for the fact that $a = (1-g)^{-1}$ and

$$\nu = \int_0^1 u_0(\eta)\eta^2\,\mathrm{d}\eta$$

is the second moment of the escape function. Therefore, one finally derives:

$$u(\eta) = \left(1 - \frac{2\nu k}{1-g}\right)u_0(\eta)\,.$$

This equation together with expression for $i(\eta)$ allows to find the constant N (see Table 6.2). Namely, we arrive at the following result:

$$N = 2\int_0^1 \mathrm{d}\eta u_0(\eta)\eta \left\{1 - \frac{2\nu k}{1-g}\right\} \left\{1 - \frac{k\eta}{1-g}\right\}$$

or

$$N = 1 - \frac{4\nu k}{1-g},$$

where we have neglected terms of the second order with respect to k. We can also write:

$$N = 1 - 4\sqrt{3}\nu s.$$

Note that functions $u(\eta)$ enter asymptotic formulas in the combination: $\Psi(\eta, \xi) = Mu(\eta)u(\xi)$. This means that one can use the following approximation, valid as $k \to 0$:

$$\Psi(\eta, \xi) = \frac{8s}{\sqrt{3}} u_0(\eta)u_0(\xi).$$

6.4.3.3 The reflection function of a semi-infinite layer

The last point in our derivations of asymptotics as $\omega_0 \to 1$ is to derive the weak absorption approximation for the reflection function of a semi-infinite medium R_∞. This will be done in two steps.

Step 1

The expression for a plane albedo of a semi-infinite medium is written by a definition as

$$r_p(\xi) = 2\int_0^1 R_\infty(\xi, \eta)\eta \, \mathrm{d}\eta.$$

We will use the following expansion of $R_\infty(\xi, \eta)$ with respect to k:

$$R_\infty(\xi, \eta) = R_{0\infty}(\xi, \eta) - kR_{1\infty}(\xi, \eta),$$

where $R_{1\infty}(\xi, \eta)$ is the function we need to find. The minus sign signifies the fact that $R_\infty(\xi, \eta) \le R_{0\infty}(\xi, \eta)$ by definition. One can see that

$$r_p(\xi) = 1 - kJ(\xi),$$

where

$$J(\xi) = 2\int_0^1 R_{1\infty}(\xi, \eta)\eta \, \mathrm{d}\eta,$$

and we used the property:

$$2\int_0^1 R_{0\infty}(\xi, \eta)\eta \, \mathrm{d}\eta = 1.$$

Step 2

We now derive the asymptotic equation for $r_p(\xi)$ as $k \to 0$ using another set of equations. This will allow us to give a relationship between $J(\xi)$ and $u_0(\xi)$. We start from the following equation (see Table 6.2):

$$i(-\xi) = 2\int_0^1 i(\eta)R_\infty(\xi,\eta)\eta \, \mathrm{d}\eta \, .$$

Substituting expansions with respect to k in this expression and ignore high-order terms, we obtain:

$$1 - \frac{k\xi}{1-g} = 2\int_0^1 \left(1 + \frac{k\eta}{1-g}\right)\left(R_{0\infty}(\xi,\eta) - kR_{1\infty}(\xi,\eta)\right)\eta \, \mathrm{d}\eta \, .$$

This means that (see Table 6.2)

$$1 - \frac{k\xi}{1-g} = 1 - kJ(\xi) + \frac{2k}{1-g}\int_0^1 R_{0\infty}(\xi,\eta)\eta^2 \, \mathrm{d}\eta$$

or

$$J(\xi) = (1-g)^{-1}\left\{\xi + 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2 \, \mathrm{d}\eta\right\} ,$$

where

$$J(\xi) = \frac{1 - r_p(\xi)}{k}$$

as was shown above. Therefore, it holds that

$$(1-g)(1-r_p(\xi))k^{-1} = 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2 \, \mathrm{d}\eta + \xi$$

or

$$r_p(\xi) = 1 - \frac{k}{1-g}\left\{\xi + 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2 \, \mathrm{d}\eta\right\} .$$

On the other hand, we have:

$$i(\xi) = Mu(\xi) + 2\int_0^1 i(-\eta)R_\infty(\xi,\eta)\eta \, \mathrm{d}\eta \, .$$

Therefore, it follows, as $k \to 0$, that

$$1 + \frac{k\xi}{1-g} = \frac{8ku_0(\xi)}{3(1-g)} + 2\int_0^1 \left(1 - \frac{k\eta}{1-g}\right)\left(R_{0\infty}(\xi,\eta) - kR_{1\infty}(\xi,\eta)\right)\eta \, \mathrm{d}\eta \, .$$

This means that

$$1 + \frac{k\xi}{1-g} = \frac{8ku_0(\xi)}{3(1-g)} + 1 - \frac{2k}{1-g}\int_0^1 R_{0\infty}(\xi,\eta)\eta^2 \, \mathrm{d}\eta - kJ(\xi)$$

or

$$\xi = \frac{8}{3}u_0(\xi) - 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2\,\mathrm{d}\eta - (1-g)J(\xi)$$

and

$$J(\xi) = (1-g)^{-1}\left\{\frac{8}{3}u_0(\xi) - 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2\,\mathrm{d}\eta - \xi\right\}.$$

Comparing this expression with the formula for $J(\xi)$ given above, we derive that

$$\frac{8u_0(\xi)}{3} = 2\xi + 4\int_0^1 R_{0\infty}(\xi,\eta)\eta^2\,\mathrm{d}\eta\,.$$

This allows us to establish the following important relationship:

$$u_0(\xi) = \frac{3}{4}\left[\xi + 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2\,\mathrm{d}\eta\right].$$

The expression in brackets is equal to $(1-g)J(\xi)$. So we have:

$$J(\xi) = \frac{4u_0(\xi)}{3(1-g)}$$

and, therefore,

$$r_p(\xi) = 1 - \frac{4ku_0(\xi)}{3(1-g)}\,.$$

The function $R_{1\infty}(\xi,\eta)$ must be symmetric with respect to the pair (ξ,η). Therefore, it follows, using the expression

$$J(\xi) = 2\int_0^1 R_{1\infty}(\xi,\eta)\eta\,\mathrm{d}\eta = \frac{4u_0(\xi)}{3(1-g)}\,,$$

that

$$R_{1\infty}(\xi,\eta) = cu_0(\xi)u_0(\eta)\,.$$

Substituting this formula in the equation given above, we derive the analytical expression for the constant c:

$$c = \frac{4}{3(1-g)}\,,$$

where we used the property

$$2\int_0^1 u_0(\xi,\eta)\eta\,\mathrm{d}\eta = 1\,.$$

Finally, we have:

$$R(\xi,\eta) = R_{0\infty}(\xi,\eta) - \frac{4k}{3(1-g)}u_0(\xi)u_0(\eta)$$

or

$$R(\xi,\eta) = R_{0\infty}(\xi,\eta) - \frac{4}{\sqrt{3}} s u_0(\xi) u_0(\eta)\,.$$

All asymptotic equations derived as $k \to 0$ are given in Table 6.3. It follows that the case of weak absorption can be studied analytically if the function $R_{0\infty}(\xi,\eta)$ is known. The escape function $u_0(\xi)$ is calculated by the integration of $R_{0\infty}(\xi,\eta)$ with respect to η as was shown above. Let us study the functions $R_{0\infty}(\xi,\eta)$ and $u_0(\xi)$ in more detail now.

Table 6.3. Asymptotic equations valid as $k \to 0 \left(\nu = \int_0^1 u_0(\xi)\xi^2\,\mathrm{d}\xi,\ g = \frac{1}{4}\int_0^\pi p(\theta)\sin 2\theta\,\mathrm{d}\theta\right)$

$R_\infty(\xi,\eta,\varphi)$	$R_{0\infty}(\xi,\eta,\varphi) - \frac{4k}{3(1-g)} u_0(\xi)u_0(\eta)$
$u(\xi)$	$\left(1 - \frac{k\nu}{2}\right) u_0(\xi)$
M	$\frac{8k}{3(1-g)}$
N	$1 - \frac{4k\nu}{1-g}$
$Mu(\xi)u(\eta)$	$\frac{8k}{3(1-g)} u_0(\xi)u_0(\eta)$
k	$\sqrt{3(1-\omega_0)(1-\omega_0 g)}$
$r_d(\xi)$	$1 - \frac{4ku_0(\xi)}{3(1-g)}$
r	$1 - \frac{4k}{3(1-g)}$

6.4.4 Nonabsorbing media

6.4.4.1 General equations

We assume that there is no absorption in a scattering medium (e.g., clouds in the visible). Then it follows, using general asymptotic equations and results presented in Table 6.3 at $\omega_0 = 1$:

$$R(\xi,\eta,\varphi) = R_{0\infty}(\xi,\eta,\varphi) - t u_0(\xi)u_0(\xi)$$

and

$$T(\xi,\eta) = t u_0(\xi)u_0(\xi)\,,$$

where

$$t = \frac{1}{0.75(1-g)\tau_0 + b}$$

and $b = 3\nu$. The plane albedo $r_p(\xi)$, the spherical albedo r, the diffuse transmittance $t_d(\xi)$ and the global transmittance t are defined in Table 6.1. We have, using results presented in Table 6.1:

$$r_p(\xi) = 1 - tu_0(\xi), \quad r = 1 - t, \quad t_d(\xi) = tu_0(\xi)$$

at $\omega_0 = 1$ and also we confirm that t coincides with the global transmittance.

It follows that the calculation of reflection and transmission functions of optically thick light scattering layers is reduced to the calculation of the reflection function of a semi-infinite layer. The function $R_{0\infty}(\xi, \eta, \varphi)$ can be used to calculate $u_0(\xi)$ and the parameter

$$\nu = \int_0^1 u_0(\xi)\xi^2 \, \mathrm{d}\xi \, .$$

Generally speaking, the functions $u_0(\xi)$ and $R_{0\infty}(\xi, \eta, \varphi)$ can be derived from the numerical solution of the corresponding integral equations (Dlugach and Yanovitskij, 1974; Sobolev, 1975; Mishchenko *et al.*, 1999). Now we introduce useful approximations for the calculation of $u_0(\xi)$, $R_{0\infty}(\xi, \eta, \varphi)$. The important property of these functions that they do not depend on the pair (ω_0, τ_0) by definition. They are completely determined by the phase function. Moreover, the dependence on the phase functions is rather weak because functions $u_0(\xi)$, $R_{0\infty}(\xi, \eta, \varphi)$ are related to the problems involving semi-infinite non-absorbing media. So multiple light scattering is quite important in this case. It leads to the averaging of the scattering features chracteristic for a single scattering event. This also means that a good starting point for the derivation of approximate solutions valid at arbitrary g is the case of $g = 0$ (e.g., isotropic scattering, $p \equiv 1$).

6.4.4.2 Auxiliary functions

We start the consideration of auxiliary functions from the well studied case of isotropic scattering. Then the nonlinear integral equation for the reflection function of a non-absorbing semi-infinite medium (de Rooij, 1985) can be presented in the following form:

$$R_{0\infty}(\xi, \eta) = \frac{1 + 2\xi \int_0^1 R_{0\infty}(\eta, \eta') \, \mathrm{d}\eta' + 2\eta \int_0^1 R_{0\infty}(\xi, \eta') \, \mathrm{d}\eta' + G(\xi, \eta)}{4(\xi + \eta)} \, ,$$

where

$$G(\xi, \eta) = 4\xi\eta \int_0^1 \int_0^1 R_{0\infty}(\xi, \eta') R_{0\infty}(\eta, \eta'') \, \mathrm{d}\eta'' \, .$$

The inspection of this equation shows that it can be reduced to the following more simple form:

$$R_{0\infty}(\xi,\eta) = \frac{H(\xi)H(\eta)}{4(\xi+\eta)}$$

with

$$H(\xi) = 1 + 2\xi \int_0^1 R_{0\infty}(\xi,\eta)\,\mathrm{d}\eta\,.$$

The last two equations allow us to formulate the integral equation for the function $H(\xi)$:

$$H(\xi) = 1 + 0.5\xi H(\xi) \int_0^1 \frac{H(\eta)}{\xi+\eta}\,\mathrm{d}\eta\,.$$

It follows immediately that $H(0) = 1.0$. Numerical calculations show that the function $H(\eta)$ can be approximated by the linear function $H(\eta) = 1 + 2\eta$. This approximation can be used as a first-guess value under the integral in the equation given above to solve the integral equation for the function $H(\xi)$ by the iteration technique. The substitution of this linear equation into the expression for $R_{0\infty}(\xi,\eta)$ presented above gives:

$$R_{0\infty}(\xi,\eta) = \frac{1 + 2(\xi+\eta) + 4\xi\eta}{4(\xi+\eta)}\,.$$

This is a rather good approximation for the isotropic scattering case. Further, we note that the value of $R_{0\infty}(\xi,\eta)$ can be separated into two parts:

$$R_{0\infty}(\xi,\eta) = R^s_{0\infty}(\xi,\eta) + R^m_{0\infty}(\xi,\eta)\,,$$

where the first term is due to single scattering $(R^s_{0\infty}(\xi,\eta) = 0.25(\xi+\eta)^{-1}$ (Chandrasekhar, 1950)) and the second term $\left(R^m_{0\infty}(\xi,\eta) = [0.5 + \xi\eta(\xi+\eta)^{-1}]\right)$ is due to multiple scattering at $p = 1$.

We make the same separation for the nonisotropic scattering case. Then, however, we have (Chandrasekhar, 1950; Kokhanovsky, 2004a):

$$R^s_{0\infty}(\xi,\eta) = 0.25p(\theta)(\xi+\eta)^{-1}$$

and we assume that it holds for multiple nonisotropic light scattering:

$$R_{0\infty}(\xi,\eta) = \frac{A + B(\xi+\eta) + C\xi\eta}{4(\xi+\eta)}\,,$$

where A, B, and C are constants to be determined. There are different ways to get these constants. In particular, integral relationships involving the function $R_{0\infty}(\xi,\eta)$ can be used (Sobolev, 1975).

Constants can be also found using the following fitting technique. The function $R_{0\infty}(\xi,\eta,\varphi)$ is calculated using the exact radiative transfer equation (see, for example, Mishchenko *et al.*, 1999) and then functions $\Xi(\xi,\eta,\varphi) = 4(\xi+\eta)\tilde{R}_{0\infty}(\xi,\eta,\varphi)$, where $\tilde{R}_{0\infty}(\xi,\eta,\varphi) = R_{0\infty}(\xi,\eta,\varphi) - R^s_{0\infty}(\xi,\eta,\varphi)$, are fitted by linear functions of the argument assuming, for example, $\eta = 1$. This technique gives: A $= 3.944$, B $= -2.5$, C $= 10.664$ for water clouds (Kokhanovsky, 2004b) and $A = 1.247$, $B = 1.186$, $C = 5.157$ for ice clouds (Kokhanovsky, 2006).

The next point is to derive the corresponding equation for the function $u_0(\xi)$. This can be done in the following way.

It was shown above that the following relationship holds:

$$u_0(\xi) = \frac{3}{4}\left[\xi + 2\int_0^1 R_{0\infty}(\xi,\eta)\eta^2\,\mathrm{d}\eta\right].$$

Let us substitute $R_{0\infty}(\xi,\eta)$ for the isotropic scattering case derived above in this equation. Then it follows that

$$u_0(\xi) = \frac{3}{4}\left[\xi + \frac{1}{2}\int_0^1 \frac{H(\xi)H(\eta)}{\xi+\eta}\eta^2\,\mathrm{d}\eta\right].$$

We substitute $H(\eta)(1-\xi(\xi+\eta)^{-1})$ for $H(\eta)\eta(\xi+\eta)^{-1}$. Then one derive:

$$u_0(\xi) = \frac{3}{4}\left[\xi + \frac{1}{2}H(\xi)\int_0^1 H(\eta)\eta\,\mathrm{d}\eta - \frac{\xi H(\xi)}{2}\int_0^1 \frac{H(\eta)}{\xi+\eta}\eta\,\mathrm{d}\eta\right].$$

This can be written as

$$u_0(\xi) = \frac{3}{4}\left[\xi + \frac{1}{2}\mathbb{C}H(\xi) - \Lambda\xi\right],$$

where

$$\mathbb{C} = \int_0^1 H(\eta)\eta\,\mathrm{d}\eta$$

and

$$\Lambda = 2\int_0^1 R_{0\infty}(\xi,\eta)\eta\,\mathrm{d}\eta\,.$$

Due to the conservation of energy law we have: $\Lambda \equiv 1$ (see Table 6.2) and, therefore,

$$u_0(\xi) = \frac{3\mathbb{C}}{8}H(\xi)\,.$$

This means that the function $u_0(\xi)$ is proportional to $H(\xi)$ at $\omega_0 = 1$. The constant $\mathbb{C}$ can easily be derived for the isotropic scattering. For this we multiply the last equation by 2ξ and integrate with respect to ξ. Then it follows that

$$\mathbb{C} = \frac{2}{\sqrt{3}}$$

where we have used the property 6.6 in Table 6.2 ($i \equiv 1$ at $\omega_0 = 1$). Therefore, we establish an important relationship:

$$u_0(\xi) = \frac{\sqrt{3}}{4}H(\xi)\,.$$

Surprisingly, two completely separate radiative transfer problems (for the determination of $H(\xi)$ and $u_0(\xi)$) have shown themselves to be interrelated. This

important theoretical result, valid for isotropic scattering, allows us to derive approximate equations for $u_0(\xi)$ using the corresponding equations for $H(\xi)$ at $\omega_0 = 1$. A number of parameterizations and approximations can be derived in such a way.

We will use the fact that $H(\xi)$ is well approximated by the function $1 + 2\xi$. Then it follows that

$$u_0(\xi) = \mathbb{Q}(1 + 2\xi) ,$$

where $\mathbb{Q} = \sqrt{3}/4 \approx 3/7$. We use the approximate equality (the error is under 1%) here to satisfy the normalization condition (property 6.6 at $i \equiv 1$ in Table 6.2). So finally, we have:

$$u_0(\xi) = \frac{3}{7}(1 + 2\xi) .$$

Although this result is strictly valid only for isotropic scattering, we find that the error of this approximation is below 2% as $\xi \geq 0.2$ for arbitrary phase functions. We also obtain that $\nu = 5/14$ and $b = 15/14 \approx 1.072$. This completes our derivations for the case $\omega_0 = 1$.

6.5 Exponential approximation

6.5.1 Semi-infinite light scattering media

Asymptotic solutions for weak absorption derived above allow us to study the influence of light absorption on radiative characteristics of turbid layers for small values of the probability of photon absorption $\beta = 1 - \omega_0$ if corresponding characteristics are known for the non-absorbing case. The results are limited to a very narrow range of β (typically, $\beta < 0.0001$). There are two possibilities for avoiding this problem. One is related to the derivation of higher-order corrections to the results given above (generally, following the same path (Minin, 1988; Yanovitskij, 1997; Melnikova and Vasyliev, 2005)).

Yet another approach is based on the exponential approximation often used in diffusion theory (Rozenberg, 1962). To demonstrate this technique, we consider the case of a semi-infinite medium. Then the spherical albedo depends on the phase function $p(\theta)$ and the single scattering albedo ω_0 only. We represent the spherical albedo as a series with respect to ω_0:

$$r(\omega_0) = \sum_{n=1}^{\infty} a_n \omega_0^n$$

with

$$r(1) = \sum_{n=1}^{\infty} a_n .$$

However, it also follows by the definition: $r(1) = 1$, which is due to the energy conservation law. Thus, one obtains that

$$\sum_{n=1}^{\infty} a_n = 1$$

and numbers a_n can be interpreted in terms of probability theory. In particular, the value of a_1 gives us the probability that a photon will be singly scattered before escaping a turbid medium. Let us substitute the following exact expansion in the expression for $r(\omega_0)$:

$$\omega_0^n \equiv (1-\beta)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \beta^j ,$$

where

$$\binom{n}{j} \equiv \frac{n!}{j!(n-j)!} .$$

Then it follows that

$$r = \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n} (-1)^j \binom{n}{j} \beta^j$$

or in the explicit form:

$$r = \sum_{n=1}^{\infty} a_n \left[1 - \beta n + \frac{\beta^2 n(n-1)}{2} - \frac{\beta^3 n(n-1)(n-2)}{6} + \dots \right] ,$$

where we accounted for equalities:

$$\binom{n}{0} = 1, \ \binom{n}{1} = n, \ \binom{n}{2} = \frac{n(n-1}{2}, \ \binom{n}{3} = \frac{n(n-1)(n-2)}{6} .$$

So we have:

$$r_s = 1 - \beta \overline{n} + \frac{\beta^2 n\overline{n(n-1)}}{2} - \frac{\beta^3 \overline{n(n-1)(n-2)}}{6} + \dots ,$$

where we used the normalization condition and defined the following averages:

$$\overline{n} = \sum_{n=1}^{\infty} n a_n, \ \overline{n(n-1)} = \sum_{n=1}^{\infty} n(n-1) a_n, \ \overline{n(n-1)(n-2)} = \sum_{n=1}^{\infty} n(n-1)(n-2) a_n$$

and so on. Here $\overline{n}$ is the average number of scattering events in the medium.

The derived expression for r is an exact formula. We have not made any approximations so far. Now we should make some assumptions so that we have the possibility of summing up the series. First of all, we assume that the value of $\overline{n}$ is large and, consequently, $\overline{n(n-1)} \approx \overline{n^2}$, $\overline{n(n-1)(n-2)} = \overline{n^3}$ and so on. Clearly, such an approximation is valid as $\beta \to 0$. This gives us:

$$r = 1 - \beta \overline{n} + \frac{\beta^2}{2} \overline{n^2} - \frac{\beta^3}{6} \overline{n^3} + \dots$$

or

$$r = \overline{\exp(-\beta n)}\,,$$

where we have used the expansion

$$\exp(-\beta n) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta n)^k}{k!}\,.$$

Thus, the value of r is given by

$$r = \sum_{n=1}^{\infty} \exp(-\beta n) a_n\,.$$

Applying the sum formula, we have:

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)\,\mathrm{d}x\,,$$

and

$$r = \int_0^{\infty} \exp(-\beta x) a(x)\,\mathrm{d}x\,.$$

One obtains, using the mean value theorem:

$$r = \exp(-\beta \overline{x})\,,$$

which is called the exponential approximation. We have also used the integral form of the normalization condition:

$$\int_0^{\infty} a(x)\,\mathrm{d}x = 1\,.$$

The problem we face now is the determination of the parameter $\overline{x}$. For this we will use the well-known asymptotic result of the radiative transfer theory derived above (see Table 6.3):

$$r = 1 - \frac{4s}{\sqrt{3}}\,,$$

which is valid as $\beta \to 0$.

The exponential approximation takes the following form as $\beta \to 0$:

$$r = 1 - \beta \overline{x}\,.$$

So, comparing these equations, we have:

$$\overline{x} = \frac{4}{k}\,,$$

where

$$k = \sqrt{3(1 - g\omega_0)\beta}$$

is the diffusion exponent of the radiative transfer theory.

Therefore, we have:

$$r = \exp\left(-\frac{4s}{\sqrt{3}}\right).$$

We notice that the combination of local optical characteristics, given by

$$y = 4\sqrt{\frac{1-\omega_0}{3(1-g\omega_0)}}$$

completely determines the spherical albedo. The value of $y = 4s/\sqrt{3}$ can be also easily measured experimentally ($y = \ln(1/r)$).

By analogy, relationships similar to the derived expression for r must be valid also for other asymptotic parameters and functions (Zege *et al.*, 1991; Kokhanovsky, 2004a). This allows us to obtain equations for the reflection function $R_\infty(\xi,\eta,\varphi)$, the plane albedo $r_{d\infty}(\xi)$, and the spherical albedo r_∞ of a semi-infinite weakly absorbing plane-parallel light scattering layer. They are shown in Table 6.4 together with correspondent equations for auxiliary constants.

Table 6.4. Asymptotic equations valid as $k \to 0$ in the framework of the exponential approximation $\left(\nu = \int_0^1 u_0(\xi)\xi^2\,\mathrm{d}\xi,\ s = \sqrt{\frac{1-\omega_0}{1-g\omega_0}}\right)$

$R_\infty(\xi,\eta,\varphi)$	$R_{0\infty}(\xi,\eta,\varphi)\exp\left\{-\frac{4su_0(\xi)u_0(\eta)}{\sqrt{3}R_{0\infty}(\xi,\eta,\varphi)}\right\}$
M	$1-\exp\left\{-\frac{8s}{\sqrt{3}}\right\}$
N	$\exp\left\{-4\sqrt{3}\nu s\right\}$
$Mu(\xi)u(\eta)$	$\left[1-\exp\left\{-\frac{8s}{\sqrt{3}}\right\}\right]u_0(\xi)u_0(\eta)$
$r_{d\infty}(\xi)$	$\exp\left\{-\frac{4su_0(\xi)}{\sqrt{3}}\right\}$
r_∞	$\exp\left\{-\frac{4s}{\sqrt{3}}\right\}$

6.5.2 Optically thick light scattering layers

The substitution of results given in Table 6.4 in general asymptotic equations presented in section 6.4.2 allows us to obtain the following analytical formula for the reflection function of an optically thick finite turbid layer valid as $\omega_0 \to 1$:

$$R(\xi,\eta,\varphi,\tau_0) = R_{0\infty}\exp(-yB(\xi,\eta,\varphi)) - te^{-x-y}u_0(\xi)u_0(\eta),$$

where we introduced a new parameter $x = k\tau$ and

$$B(\xi, \eta, \varphi) = \frac{u_0(\xi)u_0(\eta)}{R_{0\infty}(\xi, \eta, \varphi)} .$$

Also we have:

$$T(\xi, \eta) = tu_0(\xi)u_0(\eta) ,$$

where the global transmittance t is given by:

$$t = \frac{\sinh y}{\sinh(x + by)} .$$

Other radiative transfer characteristics calculated in the framework of this approximation are given in Table 6.5 (Kokhanovsky, 2004a).

Table 6.5. Radiative transfer characteristics in the framework of the exponential approximation

$$(r_{d\infty}(\xi) = \exp\left[-yu_0(\xi)\right], \ u_0(\xi) = \frac{3}{7}(1 + 2\xi),$$
$$r_\infty = e^{-y}, \ x = k\tau, \ y = k\tau, \ k = \sqrt{3(1-\omega_0)(1-g)}, \ b = 15/14 \approx 1.072).$$

Radiative characteristic	Symbol	Equation
Plane albedo	$r_d(\xi)$	$r_{d\infty}(\xi) - (r_\infty - r)u_0(\xi)$
Spherical albedor	r	$r_\infty - t\exp(-x - y)$
Diffuse transmittance	$t_d(\xi)$	$tu_0(\xi)$
Global transmittance t	$\frac{\sinh(y)}{\sinh(x+by)}$	

The exponential approximation can be used for the rapid estimations of light reflection from cloudy media and also for speeding up cloud retrieval algorithms (Kokhanovsky *et al.*, 2003). The range of the applicability of the exponential approximation with respect to smaller values of τ_0 and ω_0 can be extended using correction terms derived from the numerical solution of the radiative transfer equation. In particular, Kokhanovsky and Rozanov (2003) found that the accuracy of the exponential approximation for cloudy media can be increased using following substitutions: $B \to (1 - 0.05y)B$, $t \to t - \Delta$, where

$$\Delta = \frac{a_1 + a_2\xi\eta + a_3\xi^2\eta^2}{\tau^3} \exp(x)$$

and $a_1 = 4.86$, $a_2 = -13.08$, $a_3 = 12.76$. Therefore, the final equation for the reflection function can be written as

$$R(\xi, \eta, \varphi, \tau_0) = R_{0\infty}(\xi, \eta, \varphi) \exp(-y(1-0.05y)B(\xi, \eta, \varphi)) - (t-\Delta)e^{-x-y}u_0(\xi)u_0(\eta) .$$

This formula gives the so-called modified exponential approximation (MEA) for the reflection function of weakly absorbing optically thick turbid layers. MEA can be used at much larger values of β as compared to the linear approximation presented in section 6.4.3. We show the accuracy of MEA in Figs. 6.2 and 6.3 for the nadir observation conditions, the solar zenith angle 60° and wavelengths 550 nm and 1550 nm. It was assumed that the light scattering medium is composed of water droplets with the effective radius 4, 6, and 16 µm (Kokhanovsky and Rozanov, 2003). The gamma particle size distribution with the effective variance 38% (Kokhanovsky, 2004a) was used to model the polydispersity of droplets in clouds.

One concludes from Figs. 6.2 and 6.3 that the top-of-atmosphere reflectance over cloudy scenes can be accurately modeled in the framework of the MEA in the visible and also in the near-infrared for optically thick clouds having different microstructure. The accuracy of MEA decreases with $\beta = 1 - \omega_0$. Then general asymptotic formulas valid at arbitrary β (see section 6.4.2) must be used. However, calculations become much more involved as compared to the case of weak absorption considered in this section (King and Harshvardhan, 1986; Nakajima and King, 1992; Wauben, 1992).

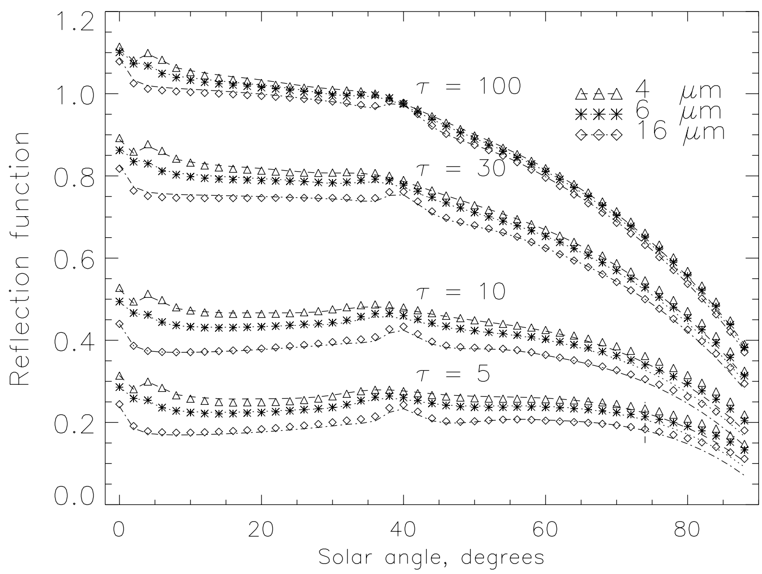

Fig. 6.2. Dependence of the reflection function of a cloud on the solar zenith angle for several values of cloud optical thickness τ and effective radius of droplets $a_{ef} = 4$, 6, and 16 µm for the nadir observation and wavelength $\lambda = 0.65$ µm (Kokhanovsky and Rozanov, 2003)

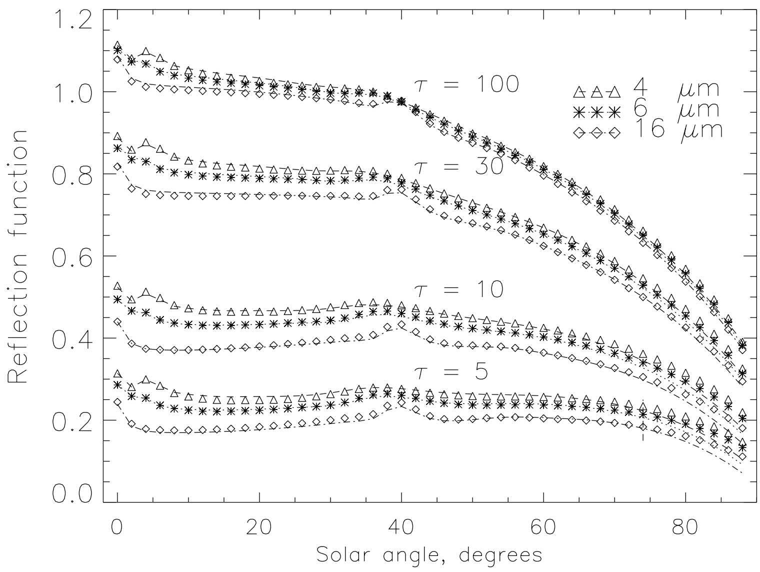

Fig. 6.3. The same as in Fig. 6.2 except at $\lambda = 1.55\,\mu\mathrm{m}$ (Kokhanovsky and Rozanov, 2003)

6.6 Conclusion

We have reviewed here the asymptotic radiative transfer theory as applied to the calculation of light reflectance and transmittance by optically thick turbid media. Derivations as presented by Sobolev (1984) are closely followed. This theory has a number of important applications in atmospheric remote sensing as described by Danielson *et al.* (1969), Zege and Katsev (1974), Melnikova and Minin (1977), Zege (1982), King (1987), Zege *et al.* (1991), Kokhanovsky *et al.* (2003), and Kokhanovsky (2004a). We hope that this review will lead to a much wider spread of this theory for the solution of direct and inverse problems of modern light scattering media optics and spectroscopy.

A historical comment must be added. Asymptotic equations for reflection and transmission functions of optically thick layers were derived for the first time by T. A. Germogenova (1961) using a different set of arguments compared to those given here. Later main equations were re-derived by van de Hulst (1968a, 1980) and also independently by Sobolev (1968, 1975, 1984). The vertical inhomogeneity of a scattering layer in the framework of ARTT was considered by Germogenova and Konovalov (1974), Ivanov (1976), Minin (1988), and Yanovitskij (1971, 1997).

The exponential approximation was initially developed by Rozenberg (1962) and improved by Bushmakova *et al.* (1971) and Zege *et al.* (1991). Kokhanovsky and Rozanov (2004a, 2004b) extended the theory to the case of radiative transfer in the gaseous absorption band. Kokhanovsky (2003) used ARTT to study light reflection and transmission by horizontally inhomogeneous turbid media.

The account for the vector properties of light has been performed by Domke (1978a, 1978b) (see also Hovenier *et al.* (2005)).

Numerical aspects of the problem were thoroughly treated by van de Hulst (1968b), Dlugach and Yanovitskij (1974), Konovalov (1974, 1975), Nakajima and King (1992), and Wauben (1992).

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